5/7/24:

Office Hours:

- Today, 1-2 pm
- Thurs, 1-2 pm

} Both on Zoom

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix}$$

Want: Basis for $\operatorname{Im} A$ & for $\operatorname{Ker} A$

(1) Row reduce A to RREF

(Row reduce A until we can identify all the pivots)

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix} \begin{matrix} \\ -2(I) \end{matrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(2) Identify all cols. w/ pivots

(3) Find basis for $\operatorname{Im} A$: All cols. in original matrix corresponding to pivots

$$\operatorname{Im} A = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

(4) Solve $A\vec{x} = \vec{0}$. So take row reduced matrix and augment with $\vec{0}$.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(5) Identify <u>free variables</u> and find the general sol. to $A\vec{x} = \vec{0}$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables

$$\underbrace{\begin{aligned} x_2 &= t \\ x_3 &= s \\ x_4 &= r \end{aligned}}$$

$t, s, r \in \mathbb{R}$

$$x_1 + 2x_2 - x_3 + 3x_4 = 0$$

$$\Rightarrow \quad x_1 = -2x_2 + x_3 - 3x_4$$

$$= -2t + s - 3r$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t + s - 3r \\ t \\ s \\ r \end{bmatrix} \overset{(6)}{=} \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} s \\ 0 \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 3r \\ 0 \\ 0 \\ r \end{bmatrix}$$

(6) Break up vector in terms of each parameter

$$= t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis vectors for Ker A

$$\text{Ker} A = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Math 33A Worksheet Week 6

TA: Emil Geisler and Caleb Partin

May 7, 2024

Exercise 1. Let $A : \mathbb{R}^4 \to \mathbb{R}^2$ be the linear transformation given by the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \end{bmatrix}$. Find a basis for $\ker A$. Find a basis for $\mathrm{Im}A$. Notice that $\dim \ker A + \dim \mathrm{Im}A = 4$.

$$\underset{e_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{e_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{e_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

Exercise 2. True or false: Explain your reasoning or find an example or counterexample.

False (a) If V is a subspace of $\mathbb{R}^3$ that does not contain any of the elementary column vectors e_1, e_2, e_3, then $V = \{\vec{0}\}$.

True (b) If v_1, v_2, v_3, v_4 are linearly independent vectors, then v_1, v_2, v_3 are linearly independent.

False (c) If v_1, v_2, v_3 are linearly independent vectors, then v_1, v_2, v_3, v_4 are linearly independent.

False (d) It is possible for a 4×4 matrix A to have $\ker A = \mathrm{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ and

not possible

$$\mathrm{Im}A = \mathrm{span}\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix} \right\rangle$$

True (e) There exists a 4×4 matrix A with $\ker A = \mathrm{span}\langle e_1, e_2, e_3 \rangle$ and $\mathrm{Im}A = \mathrm{span}\langle e_3 + e_4 \rangle$

(f) There exists a 5×5 matrix A with $\ker A = \mathrm{Im}A$. False

(g) There exists a 4×4 matrix A with $\ker A = \mathrm{Im}A$. True

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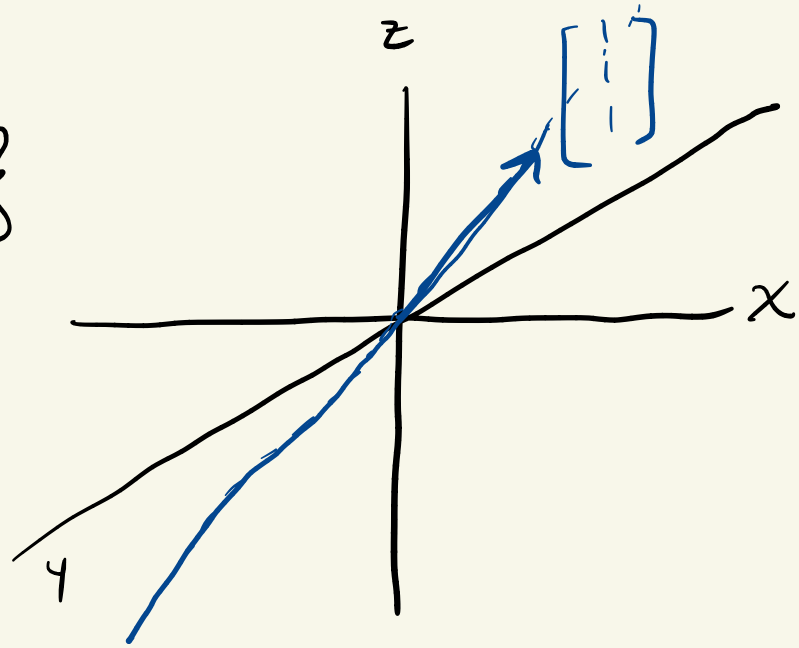
$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A \qquad \mathrm{Im}A = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

(a) Counter-example

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$



(b) $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly dependent

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = \vec{0}, \quad \text{At least one of } x_1, x_2, x_3 \text{ non-zero}$$

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 + 0 \cdot \vec{v}_4 = \vec{0}$$

$\Rightarrow \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ linearly dependent

(c) $\vec{v}_4 = \vec{v}_1 + \vec{v}_2$

(d) Rank-nullity:

$$\dim(\text{im } A) + \dim(\ker A) = \# \text{ of cols of } A$$

(e) $\ker A = \text{span}\{e_1, e_2, e_3\}$

$\text{Im } A = \underbrace{\text{span}\{e_3 + e_4\}}$ $\qquad e_3 + e_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & b & c & d \\ a & b & c & d \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & b & c & d \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$a x_1 + b x_2 + c x_3 + d x_4 = 0 \qquad x_2 = t$$

$$x_3 = s$$

$$\underline{a=1}: \quad x_1 = -b x_2 - c x_3 - d x_4 \qquad x_4 = r$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -bt - cs - dr \\ t \\ s \\ r \end{bmatrix}$$

$$= t \begin{bmatrix} -b \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -c \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -d \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Easier Way to do e:

Important: $Ae_i = i^{th}$ col. of A

So if $\ker(A) = \text{span}\{e_1, e_2, e_3\}$, then

$Ae_1 = 0$, $Ae_2 = 0$, $Ae_3 = 0$, and so

the first 3 cols. of A are 0. Combining this

with $\text{im}(A) = \text{span}\{e_3 + e_4\}$,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & a \end{bmatrix},$$

any non-zero value of a will give an answer